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Variance Measurements

Practical Use - Statistics - Long Term Prediction

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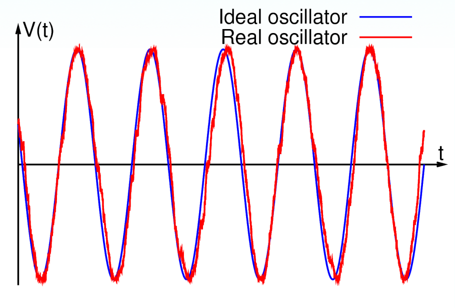
Outline

Introduction

Notations in the time domain



$$V(t) = V_0 \sin\left[2\pi\nu_0 t + \varphi(t)\right]$$

where $\varphi(t)$ is the phase "*noise*"

- **Time error $x(t)$:**

$$V(t) = V_0 \sin\left[2\pi\nu_0\left(t + x(t)\right)\right]$$

with $\quad x(t) = \dfrac{\varphi(t)}{2\pi\nu_0} \quad [s]$

"My watch is 39 seconds late":

- $t_{watch} = 10$ h 10 min 37 s
- $t_{ref} = 10$ h 11 min 16 s
- $\Rightarrow \quad x(t) = -39$ s

Frequency noise

$$V(t) = V_0 sin\left[2\pi\nu_0 t + \varphi(t)\right]$$

- **Instantaneous frequency $\nu(t)$:**

$$V(t) = V_0 \sin\left[2\pi\nu(t)\right]$$

with $$\nu(t) = \frac{1}{2\pi}\frac{d\left[2\pi\nu_0 t + \varphi(t)\right]}{dt} = \nu_0 + \frac{1}{2\pi}\frac{d\varphi(t)}{dt} \qquad [Hz]$$

- **Frequency noise $\Delta\nu(t)$:**

$$\Delta\nu(t) = \frac{1}{2\pi}\frac{d\varphi(t)}{dt} \qquad [Hz]$$

- **Frequency deviation $y(t)$:**

$$y(t) = \frac{\Delta\nu(t)}{\nu_0} = \frac{1}{2\pi\nu_0}\frac{d\varphi(t)}{dt} \qquad [dimensionless]$$

Frequency noise *vs* Phase noise

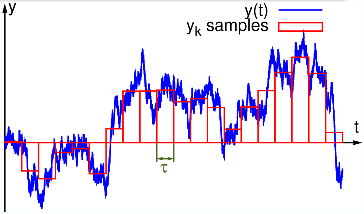
Phase and frequency noise: 2 representations of 1 phenomenon

$$\left.\begin{array}{l} x(t) = \dfrac{\varphi(t)}{2\pi\nu_0} \\[2ex] y(t) = \dfrac{1}{2\pi\nu_0}\dfrac{d\varphi(t)}{dt} \end{array}\right\} \Rightarrow \quad y(t) = \frac{dx(t)}{dt}$$

A fundamental difference:

- $\varphi(t)$ and $x(t)$ are instantaneous
- $\Delta\nu(t)$ and $y(t)$ have to be averaged

$$\bar{y}_k = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)dt = \frac{x(t_k+\tau) - x(t_k)}{\tau}$$



Notations in the frequency domain

Power Spectral Densities (PSD)

- Fourier Transform (finite energy):

$$\Phi(f) = \int_{-\infty}^{+\infty} \varphi(t) e^{-j2\pi f t} dt \qquad [s]$$

- Energy Spectral Density (finite energy):

$$|\Phi(f)|^2 = \left| \int_{-\infty}^{+\infty} \varphi(t) e^{-j2\pi f t} dt \right|^2 \qquad [s^2]$$

- Power Spectral Density (finite power):

$$S_\varphi(f) = \left\langle \lim_{T\to\infty} \left[\frac{1}{T} \left| \int_{-T/2}^{+T/2} \varphi(t) e^{-j2\pi f t} dt \right|^2 \right] \right\rangle \qquad [s] \equiv [Hz^{-1}]$$

Relationships between PSD

Time error PSD: $S_x(f)$

- $x(t) = \frac{\varphi(t)}{2\pi\nu_0}$ $\Rightarrow$ $S_x(f) = \frac{1}{4\pi^2\nu_0^2} S_\varphi(f)$
- Dimension: $[s^3] \equiv [Hz^{-3}]$

Frequency deviation PSD: $S_y(f)$

- $y(t) = \frac{1}{2\pi\nu_0}\frac{d\varphi(t)}{dt}$ $\Rightarrow$ $S_y(f) = \frac{f^2}{\nu_0^2} S_\varphi(f)$
- $y(t) = \frac{dx(t)}{dt}$ $\Rightarrow$ $S_y(f) = 4\pi^2 f^2 S_x(f)$
- Dimension: $[s] \equiv [Hz^{-1}]$

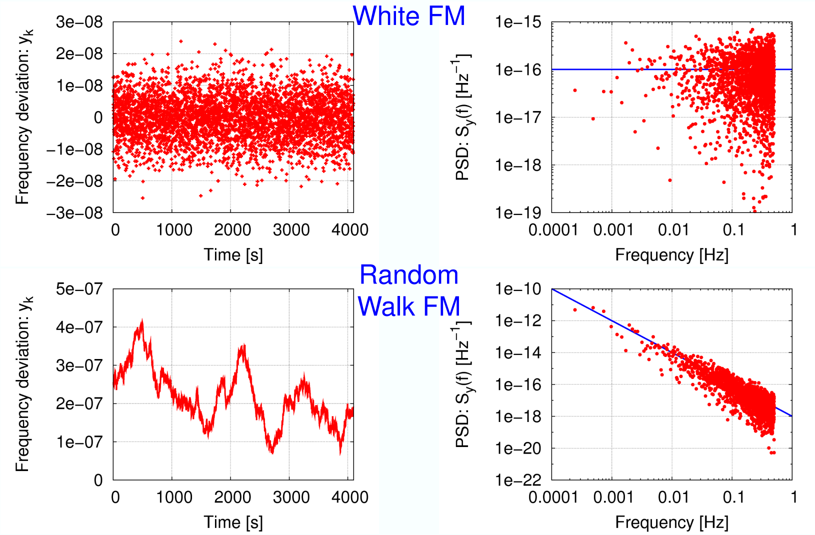
Noise model

The power law noise model

$$S_y(f) = \sum_{\alpha=-2}^{+2} h_\alpha f^\alpha \qquad \alpha \text{ integer}$$

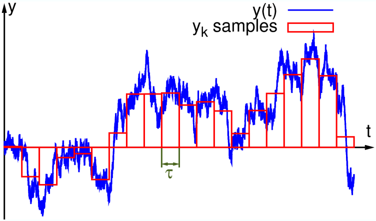
$S_y(f)$	$S_\varphi(f)$	Noise type	Origin
$h_{-2}f^{-2}$	$b_{-4}f^{-4}$	Random Walk Freq. Mod.	Environment
$h_{-1}f^{-1}$	$b_{-3}f^{-3}$	Flicker F.M.	Resonator
h_0	$b_{-2}f^{-2}$	White F.M.	Thermal noise
$h_1 f$	$b_{-1}f^{-1}$	Flicker Phase Mod.	Electronic noise
$h_2 f^2$	b_0	White P.M.	External white noise

White FM *vs* Random Walk FM



A statistical estimator
as well as a spectral analysis tool

- Definition of the true variance:
  $$I^2(\tau) = \left\langle \left(\bar{y}_k - \langle \bar{y}_k \rangle\right)^2 \right\rangle$$
- Estimation of the true variance:
  $$\sigma^2(N,\tau) = \frac{1}{N-1}\sum_{i=1}^{N}\left(\bar{y}_i - \frac{1}{N}\sum_{j=1}^{N}\bar{y}_j\right)^2$$



- **The Allan variance (2-sample variance):**
  $$\sigma_y^2(\tau) = \langle \sigma^2(2,\tau) \rangle = \left\langle \sum_{i=1}^{2}\left(\bar{y}_i - \frac{1}{2}\sum_{j=1}^{2}\bar{y}_j\right)^2 \right\rangle$$
  $$\boldsymbol{\sigma_y^2(\tau) = \frac{1}{2}\left\langle (\bar{y}_2 - \bar{y}_1)^2 \right\rangle} = \mathrm{AVAR}(\tau)$$

$\langle\rangle$ stands for:

- ensemble average
- time average
- $\equiv$ convolution...

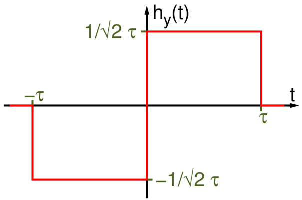
A spectral analysis tool

as well as a statistical estimator

Convolution in the time domain…

$$\sigma_y^2(\tau) = \left\langle \left[\int_{-\infty}^{+\infty} y(t) h_y(t_k - t) dt \right]^2 \right\rangle$$

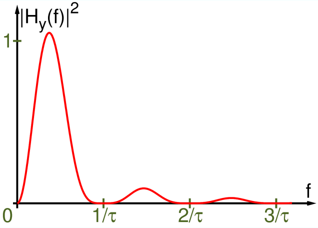
with
$$\begin{cases} h_y(t) = \dfrac{-1}{\sqrt{2\tau}} & \text{if} \quad -\tau \geq t < 0 \\ h_y(t) = \dfrac{+1}{\sqrt{2\tau}} & \text{if} \quad 0 \geq t < \tau \\ h_y(t) = 0 & \text{else} \end{cases}$$



…filtering in the frequency domain

$$\sigma_y^2(\tau) = \int_0^{\infty} S_y(f) \left| H_y(f) \right|^2 df$$

with $|H_y(f)|^2 = |\mathrm{FT}\,[h_y(t)]|^2 = 2 \dfrac{\sin^4(\pi\tau f)}{(\pi\tau f)^2}$



Convergence criterion: the moment condition

Convergence for drift

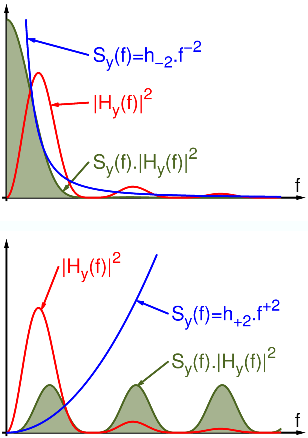
$\sigma_y^2(\tau)$ is a first-order difference (derivative):

- not sensitive to constant (lin. ph. drift)
- sensitive to linear frequency drift

Convergence for power-law noise

$$\sigma_y^2(\tau) = \int_0^{\infty} h_\alpha f^\alpha \left|H_y(f)\right|^2 df$$

- converges for f^{-2}, f^{-1} and white FM
- does not converge for f^1 and f^2 FM



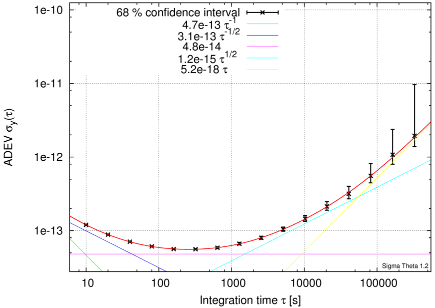
The moment condition

$$\int_{-\infty}^{+\infty} \left|H_y(f)\right|^2 f^\alpha df \text{ converges} \Leftrightarrow \int_{-\infty}^{+\infty} h_y(t) t^q dt = 0 \quad \text{for } 0 \le q \le \frac{-\alpha - 1}{2}$$

Link between noise levels and variance responses

$$\sigma_y^2(\tau) = 2\int_0^{+\infty} h_\alpha f^\alpha \frac{\sin^4(\pi\tau f)}{(\pi\tau f)^2} df$$

f_h is the high cut-off frequency



$S_y(f)$	$h_{-2}f^{-2}$	$h_{-1}f^{-1}$	$h_0 f^0$	$h_{+1}f^{+1}$	$h_{+2}f^{+2}$
$\sigma_y^2(\tau)$	$\frac{2\pi^2 h_{-2}\tau}{3}$	$2\ln(2)h_{-1}$	$\frac{h_0}{2\tau}$	$\frac{[1.04 + 3\ln(2\pi f_h\tau)]\, h_{+1}}{4\pi^2\tau^2}$	$\frac{3h_{+2}f_h}{4\pi^2\tau^2}$

Practical calculation of the Allan variance

Calculation from time error samples

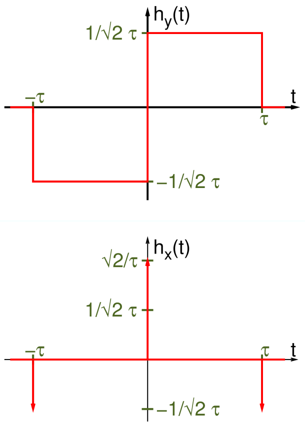
Calculation from frequency deviation

$$\sigma_y^2(\tau) = \frac{1}{2}\left\langle (\bar{y}_2 - \bar{y}_1)^2 \right\rangle = \left\langle [y(t) * h_y(t)]^2 \right\rangle$$

$$\sigma_y^2(\tau) = \int_0^\infty S_y(f)\,|H_y(f)|^2\, df$$

Calculation from time error samples

- $\sigma_y^2(\tau) = \int_0^\infty S_x(f)\,|j2\pi f H_y(f)|^2\, df$
  $= \left\langle [x(t) * h_x(t)]^2 \right\rangle \quad \text{with } h_x(t) = \dfrac{dh_y(t)}{dt}$
- $\sigma_y^2(\tau) = \frac{1}{2}\left\langle (\bar{y}_2 - \bar{y}_1)^2 \right\rangle$
  $= \dfrac{1}{2\tau}\left\langle [x(t+\tau) - 2x(t) + x(t-\tau)]^2 \right\rangle$



Practical calculation of the Allan variance

Calculation from spectral density

Calculation from frequency deviation

$$\sigma_y^2(\tau) = \frac{1}{2}\left\langle (\bar{y}_2 - \bar{y}_1)^2 \right\rangle = \int_0^{\infty} S_y(f)\,|H_y(f)|^2\, df$$

Calculation from spectral density

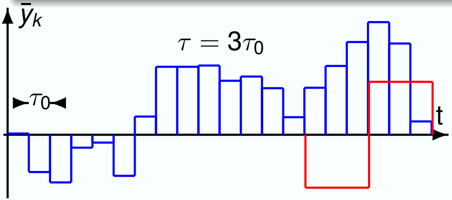
From a Phase Noise Measurement System:
$S_y(f_k)$ with $f_k \in \{f_1, 2f_1, \ldots, kf_1, \ldots, Nf_1\}$

$$\sigma_y^2(\tau) = 2\sum_{k=1}^{N} S_y(kf_1)\frac{\sin^4(\pi\tau kf_1)}{(\pi\tau kf_1)^2} f_1$$

f_h is the bandwidth of the system

Allan variance with or without overlapping

Allan variance with overlapping

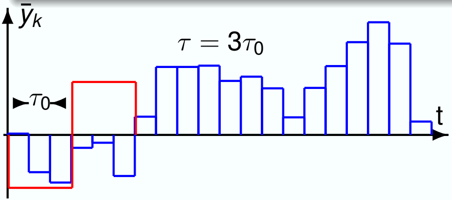


τ_0-steps moving average

Benefits and drawbacks :

- lower dispersion
- more correlated estimates

Allan variance without overlapping



Shifted by τ-steps :
$\tau = 3\tau_0 \Leftrightarrow \bar{Y}_1 = (\bar{y}_1 + \bar{y}_2 + \bar{y}_3)/3$

Benefits and drawbacks :

- less correlated estimates
- higher dispersion

Allan variance versus Allan deviation

$$\mathrm{ADEV}(\tau) = \sigma_y(\tau) = \sqrt{\sigma_y^2(\tau)}$$

Physical meaning

- $\sigma_y(\tau) \equiv \dfrac{\Delta t}{\tau}$
  Ex.: Cs clock $\quad \sigma_y(\tau = 1\mathrm{day}) = 10^{-14}$
  $\Rightarrow \quad \Delta t \approx 10^{-14} \cdot 10^5 = 10^{-9} =$ 1 ns over 1 day
- $\sigma_y(\tau) \equiv \dfrac{\Delta f}{\nu_0}$ (during τ)
  Ex.: H-Maser @ 100 MHz $\quad \sigma_y(\tau = 1\mathrm{hour}) = 10^{-14}$
  $\Rightarrow \quad \Delta f \approx 10^{-14} \cdot 10^8 = 10^{-6} = 1\mu$Hz over 1 hour

Benefits and drawbacks

- Easy to interpret
- Biased

Chi-squared and Rayleigh distribution

Allan variance: $\sigma_y^2(\tau) = \frac{1}{2}\left\langle(\bar{y}_2 - \bar{y}_1)^2\right\rangle$

Estimate: $\hat{\sigma}_y^2(\tau) = \frac{1}{2N}\sum_{i=1}^{N}(\bar{y}_2 - \bar{y}_1)^2$

- $\bar{y}_2 - \bar{y}_1$: Gaussian centered values
- $(\bar{y}_2 - \bar{y}_1)^2$: χ_1^2 distribution
- $\frac{1}{2N}\sum_{i=1}^{N}(\bar{y}_2 - \bar{y}_1)^2$: χ_N^2 distribution

Allan deviation: $\sigma_y(\tau) = \sqrt{\frac{1}{2}\left\langle(\bar{y}_2 - \bar{y}_1)^2\right\rangle}$

Estimate: $\hat{\sigma}_y(\tau) = \sqrt{\frac{1}{2N}\sum_{i=1}^{N}(\bar{y}_2 - \bar{y}_1)^2} \quad \Rightarrow \chi_N$ distributed (Rayleigh)

***N* is the number of Equivalent Degrees of Freedom (EDF)**

Reminder of the Equivalent Degrees of Freedom

Meaning of the EDF

$\mathrm{Mean}(\chi^2_\nu) = \nu$ and $\mathrm{Variance}(\chi^2_\nu) = 2\nu$

The EDF ν contains the information about the dispersion of the random variable χ^2_ν

Estimation of the EDF

$$\hat{\sigma}^2_y(\tau) = \frac{1}{2N}\sum_{i=1}^{N}(\bar{y}_2 - \bar{y}_1)^2 \quad \Rightarrow \quad \chi^2_N \text{ if } \{\bar{y}_1, \bar{y}_2, \ldots\} \text{ uncorrelated!}$$

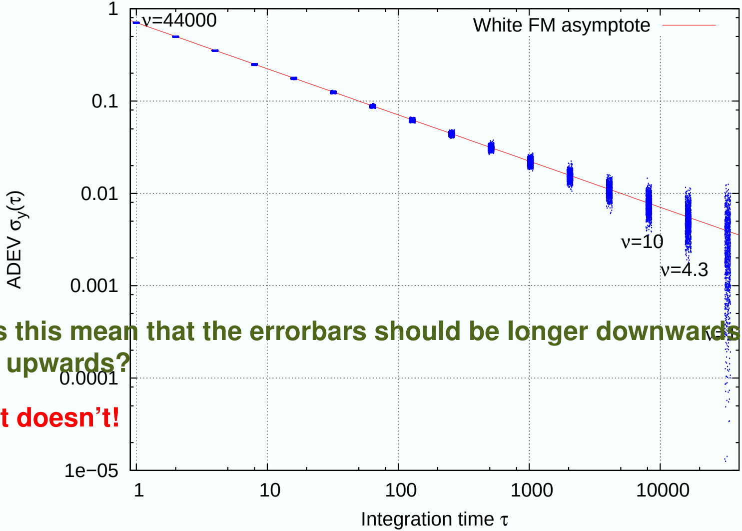
False:

- for low frequency noises (flicker and random walk FM)
- with overlapping variances

Algorithm for estimating the EDF:

- **C. Greenhall and W. Riley, 2003**, *"Uncertainty of Stability Variances Based on Finite Differences"* (35$^{\text{th}}$ PTTI). Used in *Stable 32* as well as in *SigmaTheta*.

Dispersion of Allan deviation estimates



Does this mean that the errorbars should be longer downwards than upwards?

No, it doesn't!

World of the model versus world of measures

- θ is the model parameter
- ξ is a measure of the parameter

Example:

Parameter $\sigma_y(\tau = 10 \text{ s}) = \sqrt{h_0/20}$ where h_0 is the white FM level

Measure $\hat{\sigma}_y(\tau)$ is a measure of $\sigma_y(\tau = 10 \text{ s})$

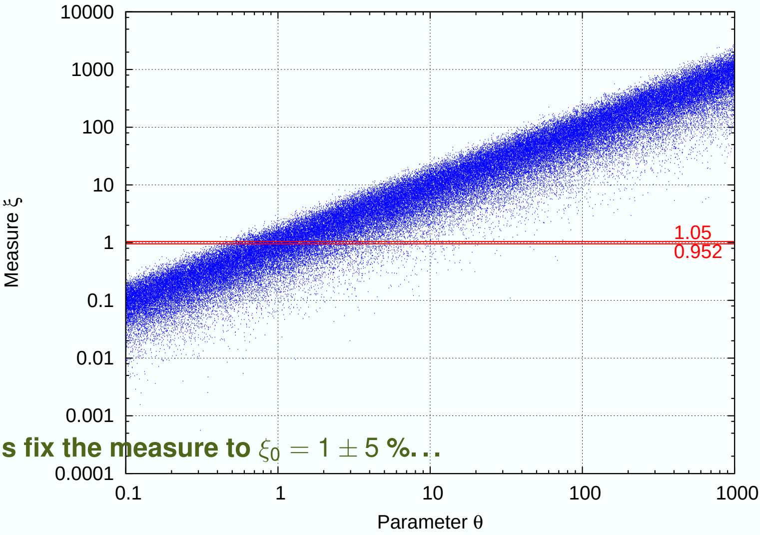
World of the model (direct problem):

Knowing the parameter θ_0, how is the measure ξ distributed?
Only valid for simulations!

World of the measures (inverse problem):

Knowing the measure ξ_0, how to estimate a confidence interval over θ?
It's the right question of the metrologist!

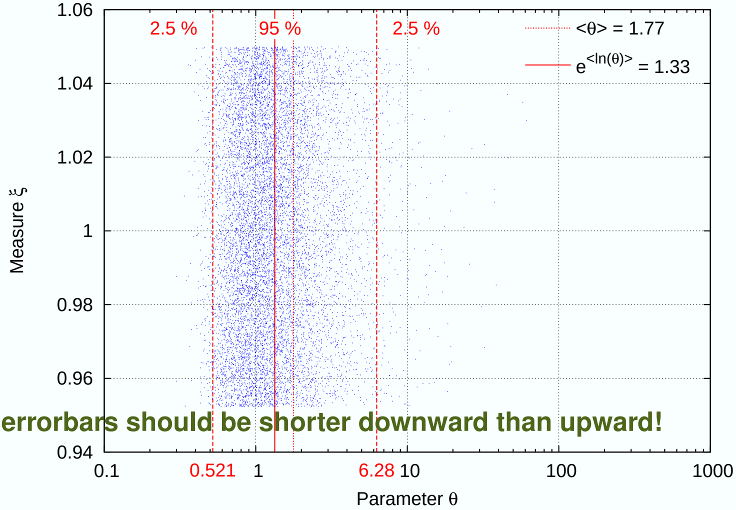
Model parameter and measure for a χ_2 distribution



Let us fix the measure to $\xi_0 = 1 \pm 5$ %...

Model parameter values for a measure $\xi_0 \approx 1$

Theoretical results versus 20,000 simulations



The errorbars should be shorter downward than upward!

Study of a χ distribution with 2 degrees of freedom

Direct problem

- Probability density function: $p(\chi) = \chi e^{-\chi^2/2}$
- The pdf is normalized: $\int_0^\infty p(\chi) d\chi = 1$
- Mathematical expectation: $\mu = \int_0^\infty \chi \cdot p(\chi) d\chi = \sqrt{\frac{\pi}{2}}$
- Cumulative distribution function: $P(\chi) = \int_0^\chi p(y) dy = 1 - e^{-\chi^2/2}$
- Inverse cdf: $P^{-1}(\alpha) = \sqrt{-2\ln(1-\alpha)}$

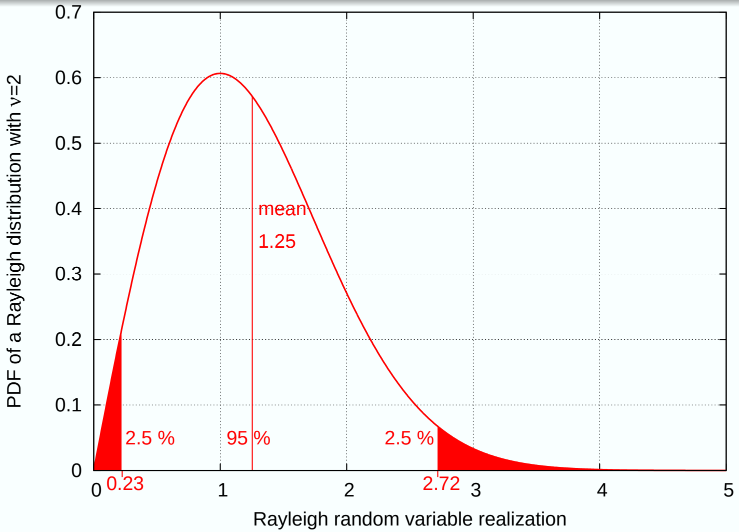
Confidence Interval of a χ_2 random variable

- $\{\ldots \chi_i \ldots\}$ is a set of realizations of the random variable χ
  - $P^{-1}(0.025) \approx 0.22502$
    $\Rightarrow \quad \chi_i < 0.22502$ with 2.5% confidence
  - $P^{-1}(0.975) \approx 2.7162$
    $\Rightarrow \quad \chi_i < 2.7162$ with 97.5% confidence

- **Confidence Interval:**
  - $E(\chi) \approx 1.2533$
  - $0.22502 < \chi_i < 2.7162$ with 95% confidence

- **General case** of a random variable $x = k \cdot \chi$
  - Estimation of the scale factor: $k = \dfrac{E(x)}{E(\chi)} \approx \dfrac{< x >}{\mu}$
  - $\Rightarrow \quad 0.22502 \cdot k < x_i < 2.7162 \cdot k$ with 95% confidence

Probability density function of a χ_2 distribution



Conditionnal probabilities

Reduced variable (I)

- Let us consider the standard χ_2^2 variable: $\chi_2^2 = X_1^2 + X_2^2$ where X_1 and X_2 are 2 Gaussian centered standard random variables

$$\Rightarrow \quad E(\chi_2^2) = 2 \quad \Rightarrow \quad E\left(\frac{1}{2}\chi_2^2\right) = 1.$$

- We assume that $\hat{\sigma}_y^2(\tau) = \xi^2$ is χ_2^2 distributed and is an unbiased estimator of the parameter $\sigma_y^2(\tau) = \theta^2$:

$$E\left(\frac{\xi^2}{\theta^2}\right) = 1.$$

- We can then define the reduced variable χ_2^2 as:

$$\chi_2^2 = 2\frac{\xi}{\theta}.$$

Reduced variable (II)

- By extension, we assume that $\hat{\sigma}_y(\tau) = \xi$ is χ_2 distributed and ξ is an estimator of the parameter $\sigma_y(\tau) = \theta$.
- We can then define the reduced variable χ as:

$$\chi = \sqrt{2}\frac{\xi}{\theta}.$$

- The differential $d\chi$ is then:

$$d\chi = \frac{\partial \chi}{\partial \xi} d\xi + \frac{\partial \chi}{\partial \theta} d\theta.$$

- From $p(\chi)$ we can deduce $P(\xi|\theta_0)$ and $P(\theta|\xi_0)$

Parameter estimation from a single measure

Usual frequentist reasonning

We assume that the measure ξ represents the estimate $\hat{\sigma}_y(\tau)$ and the parameter θ stands for the real unknown value $\sigma_y(\tau)$.

- **Reduced variable:** $\chi = \sqrt{2}\xi/\theta$
- **Low bound:** $B_{2.5\%} \approx 0.22502$
- **High bound:** $B_{97.5\%} \approx 2.7162$
- **95 % confidence interval:** $0.22502 < \sqrt{2}\xi/\theta < 2.7162$
- **Frequentist reversal:** $\dfrac{\sqrt{2}\xi_0}{2.7162} < \theta < \dfrac{\sqrt{2}\xi_0}{0.22502}$ @ 95 %

$\Rightarrow$ $\mathbf{0.52066 \cdot \xi_0 < \theta < 6.2847 \cdot \xi_0}$ **with 95 % confidence.**

We obtain directly the same result from $P(\theta|\xi_0)$ (as well as from the Bayesian method with a total lack of knowledge prior).

Generalization to a χ_ν distribution

- **Reduced variable:** $\chi = \sqrt{\nu}\frac{\xi}{\theta}$
- **pdf:** $p(\chi) = \frac{2^{1-\nu/2}\chi^{\nu-1}e^{-\chi^2/2}}{\Gamma(\nu/2)}$
- **Mathematical expectation:** $\mu_\nu = \sqrt{2}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$
- **cdf:** $P(\chi) = \Gamma\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right)$

Parameter estimation

Allan variance

- We assume that $\hat{\sigma}_y^2(\tau) = \xi^2$ is χ_2^2 distributed and is an estimator of the parameter $\sigma_y^2(\tau) = \theta^2$.
- **Reduced variable:** $\chi_2^2 = 2\dfrac{\xi^2}{\theta^2}$
- **Mathematical expectation:** $\langle \chi_2^2 \rangle = 2$

  $$\Rightarrow \quad \left\langle 2\frac{\xi^2}{\theta^2} \right\rangle = 2 \quad \Leftrightarrow \quad \left\langle \frac{\xi^2}{\theta^2} \right\rangle = 1$$

- **For a given parameter θ_0^2:** $\langle \xi^2 \rangle = \theta_0^2$
  The average of the measures given by the parameter θ_0^2 is equal to θ_0^2: ξ^2 **is an unbiased estimator of** θ_0^2.
- **For a given measure ξ_0^2:** $\langle \theta^2 \rangle = \xi_0^2$
  The average of the parameter values which give the measure ξ_0^2 is equal to ξ_0^2: **the measure ξ_0^2 may be used for representing the parameter θ^2** (for fitting... except in a log-log plot!)

Parameter estimation

Allan deviation

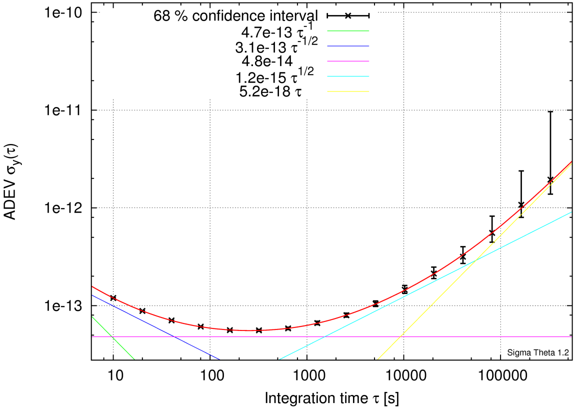
- We assume that $\hat{\sigma}_y(\tau) = \xi$ is χ_2 distributed and is an estimator of the parameter $\sigma_y(\tau) = \theta$.
- **Reduced variable:** $\chi_2^2 = \sqrt{2}\frac{\xi}{\theta}$
- **Mathematical expectation:** $\langle \chi_2^2 \rangle = \mu = \sqrt{\pi/2}$

$$\Rightarrow \quad \left\langle \sqrt{2}\frac{\xi}{\theta} \right\rangle = \sqrt{\frac{\pi}{2}} \quad \Leftrightarrow \quad \left\langle \frac{\xi}{\theta} \right\rangle = \sqrt{\frac{\pi}{4}}$$

- **For a given parameter** θ_0**:** $\langle \xi \rangle = \sqrt{\pi/4}\theta_0 \approx 1.128\theta_0$
  ξ is a biased estimator of θ_0 (overestimated by 13%).
- **For a given measure** ξ_0**:** $\langle \theta \rangle = \sqrt{4/\pi}\xi_0 \approx 0.886\theta_0$
  the measure ξ_0 should NOT be used for representing the parameter θ (underestimated by 13 %).

Never fit the curve of Allan deviation, always use the Allan variance!

Increasing the number of edf: the Total variance



The longer the time duration, the larger the uncertainty.

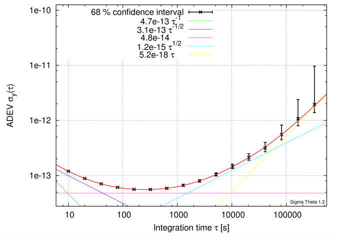
What about very long term stability ?

In order to improve estimates for very long term, D. Howe developed:

- Total variance: *UFFC-47(5), 1102-1110 (2000)*
- $\widehat{\text{Theo}}$: *Metrologia 43, S322-S331 (2006)*

Fitting curve over variance measurement (I)

$$\sigma_y^2(\tau) = \sum_{i=0}^{4} C_i \Phi_i(\tau) \quad \text{with} \quad \Phi_i(\tau) = \tau^{i-2}$$



How to estimate the C_i coefficients?

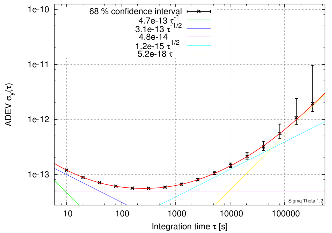
Classical least squares:

$$\sum_{j=1}^{N} \left(\hat{\sigma}_y^2(\tau_j) - \sum_{i=0}^{4} C_i \Phi_i(\tau_j) \right)^2 \quad \text{is minimum}$$

- not suitable for high dynamic
- not suitable for positive or null values
- not suitable for variance curves

Fitting curve over variance measurement (II)

$$\sigma_y^2(\tau) = \sum_{i=0}^{4} C_i \Phi_i(\tau) \quad \text{with} \quad \Phi_i(\tau) = \tau^{i-2}$$



How to estimate the C_i coefficients?

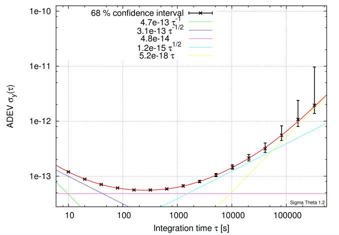
Relative least squares:

$$\sum_{j=1}^{N} \left[\frac{1}{\hat{\sigma}_y^2(\tau_j)} \left(\hat{\sigma}_y^2(\tau_j) - \sum_{i=0}^{4} C_i \Phi_i(\tau_j) \right) \right]^2 \quad \text{is minimum}$$

- equivalent to a least square fit on log-log plot
- doesn't take into account the uncertainties over the Allan variance measures
- not suitable for variance curves

Fitting curve over variance measurement (III)

$$\sigma_y^2(\tau) = \sum_{i=0}^{4} C_i \Phi_i(\tau) \quad \text{with} \quad \Phi_i(\tau) = \tau^{i-2}$$



How to estimate the C_i coefficients?

Weighted relative least squares:

$$\sum_{j=1}^{N} \left[\frac{1}{\mathrm{EDF}\left[\hat{\sigma}_y^2(\tau_j)\right]} \frac{1}{\hat{\sigma}_y^2(\tau_j)} \left(\hat{\sigma}_y^2(\tau_j) - \sum_{i=0}^{4} C_i \Phi_i(\tau_j) \right) \right]^2 \quad \text{is minimum}$$

- equivalent to a least square fit on log-log plot
- takes into account the uncertainties over the Allan variance measures
- suitable for variance curves

Estimation of the noise levels from the fitting curve

$$\sigma_y^2(\tau) = \sum_{i=0}^{4} C_i \Phi_i(\tau) \quad \text{with} \quad \Phi_i(\tau) = \tau^{i-2}$$

- $C_0\tau^{-2}$ **White or Flicker PM:** $h_{+2} = \dfrac{4\pi^2 C_0}{3 f_h}$ or $h_{+1} \approx 4\pi^2 C_0$
- $C_1\tau^{-1}$ **White FM:** $h_0 = 2C_1$
- $C_2\tau^0$ **Flicker FM:** $h_{-1} = \dfrac{C_2}{2\ln(2)}$
- $C_3\tau$ **Random Walk FM:** $h_{-2} = \dfrac{3C_3}{2\pi^2}$
- $C_4\tau^2$ **Linear frequency drift:** $D_1 = \sqrt{2C_4}$

Uncertainties Δh_α ? See *Vernotte et al., IM-42(2), 342-350 (1993)*

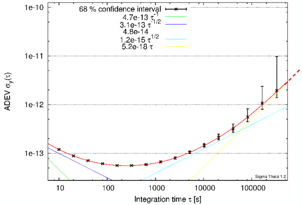
Extrapolation to very long term time stability

Some recommendations

Is it possible to extrapolate the fit beyond the last Allan variance measure?

Sometimes yes, but very carefully !

We ought already to answer to the following questions…



1. Is the longest term noise or drift asymptote visible on the curve?
   Flicker FM for Cesium, random walk FM and/or linear frequency drift otherwise
2. Is this asymptote well determined ?
   This asymptote must be dominant for at least 2-3 octaves
3. Is the curve compatible with a null coefficient for the longest term noise or drift ?
   The bottom uncertainty domains can fit correctly the other asymptotes

If you answered YES to the questions 1 and 2, and NO to the last question, you may try…